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THE MECHANISM OF THE HARD APPEARANCE OF A TWO-FREQUENCY OSCILLATION MODE IN THE CASE OF ANDRONOV-HOPF REVERSE BIFURCATION*

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The mapping of Poincaré secants is used to prove that a two-frequency oscillation mode (2-torus) can arise as a result of the hard loss of stability of the equilibrium state. A necessary condition for the transition is the presence close to the equilibrium state of a saddle periodic motion, the unstable manifold of which is attracted to the stationary manifold. At the instant when the cycle vanishes (Andronov-Hopf reverse bifurcation) a close-to-homoclinic situation arise, when the unstable separatrix of the stationary state returns to a small neighbourhood of it along a stable direction.

Sufficient conditions are found for the Poincaré mapping to have an invariant curve corresponding to the appearance of a 2-torus in the initial system of differential equations. The possible connection of this scenario of stationary state with torus transition with the observed /1, 2/ mixed convection in a vertical layer with wavy boundaries in the case of numerical simulation is discussed.

1. Formulation of the problem. We consider the system of differential equations

$$u^{\cdot} = F(u, \mu), \quad u \in R^n, \quad \mu \in [-\mu_0, \mu_0] \tag{1.1}$$

where F is a C^∞-smooth or analytic function of u, μ. We assume that $F(0, 0) = 0$ and that, when the sign of μ changes, an Andronov-Hopf reverse bifurcation occurs in the system. Let the equilibrium state O at $\mu = 0$ be a node with respect to the hyperbolic variable and an unstable non-hyperbolic focus in the central manifold.

In the simplest case $n = 3$, when there is just one hyperbolic variable x, a smooth replacement of the coordinates and time can be used in some domain of variables μ and u, where $|\mu|$ and $|u|$ are sufficiently small, to reduce system (1.1) to the form

$$\begin{aligned} \rho^{\cdot} &= \mu\rho + \rho^3 + a\rho^5, \quad \varphi^{\cdot} = \omega \\ x^{\cdot} &= -\lambda x + N(\rho, \varphi, x, \mu) \end{aligned} \tag{1.2}$$

where ρ and φ are polar coordinates in the central manifold; the function N includes higher-order terms, and $N = 0$ for $x = 0$.

For $\mu > 0$ the system has an equilibrium state (CP) of saddle type. If $\mu < 0$ there is a stable CP and a saddle periodic motion L_μ branching from it at the point $\mu = 0$. Let $W_0^s(\mu)$

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and $W_0^u(\mu)$ denote respectively the stable and unstable sets (separatrices) of the points O with $\mu \geqslant 0$, and $W_L^s(\mu)$, $W_L^u(\mu)$ the stable and unstable manifolds of the periodic motion with $\mu < 0$. Clearly, $\dim W_0^u = 2$, $\dim W_0^s = 1$ when $\mu \geqslant 0$, and $\dim W_L^u = 2$, $\dim W_L^s = 2$ when $\mu < 0$.

The basic assumption about the non-local behaviour of the trajectories is that, when $\mu = 0$, the unstable separatrix CP returns to a small neighbourhood of its stable set (a situation similar to the formation of a homoclinic). The mathematical statement of this fact is given below. Notice that the conditions imposed are of a general type and do not increase the codimensionality of the bifurcation.

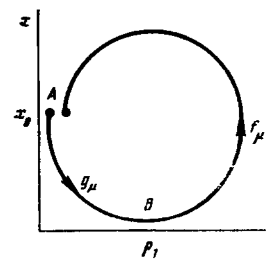

Fig.1

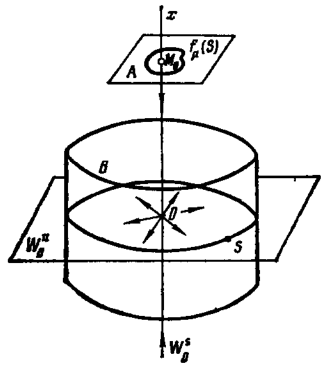

Fig.2

2. Non-local mapping. Consider the behaviour of the trajectories that issue from a small neighbourhood of CP O. Following the approach used in /3, 4/, we construct the Poincaré mapping as the superposition of a local mapping (with respect to the trajectories close to CP) and a global mapping (with respect to the trajectories that travel into the neighbourhood of a non-local piece of the unstable separatrix).

We take two areas transverse to the trajectories: $A = \{x = x_0, \rho \leqslant \varepsilon_0\}$, $B = \{\rho = \rho_1, |x| < \varepsilon_1\}$, where $\varepsilon_1 < x_0$, $\varepsilon_0 < \rho_1$ and fairly small positive numbers (Figs.1 and 2). The trace of $W_0^u(0)$ on B can be written as $x = 0$ (the circle S), and the trace of $W_0^s(0)$ on A as $\rho = 0$ (the point M_0). We shall assume that any semitrajectory that issues from S reaches A. This means that a mapping f_μ is defined with respect to the trajectories of system (1.1) from a small neighbourhood of S on B into A, which is a diffeomorphism both for $\mu = 0$, and for sufficiently small $\mu > 0$ (we assume that ε_1 is so small that the diffeomorphism f_μ is defined for all points of B).

The closed curve $f_\mu(S)$ may (case a, Fig.2) or may not (case b) embrace the point M_0. We write the mapping

$$
\begin{aligned}
&f_\mu: \quad (x_1, \varphi_1) \mapsto (\bar\rho_0, \bar\varphi_0) \\
&\bar\rho_0 = R(\varphi_1, \mu) + G(\varphi_1, x_1, \mu)\, x_1 \\
&\bar\varphi_0 = \Gamma\varphi_1 + P(\varphi_1, \mu) + V(\varphi_1, x_1, \mu)\, x_1 \\
&R(\varphi_1, 0) \geqslant R_L(0) > 0, \quad dR_L/d\mu > 0
\end{aligned}
\tag{2.1}
$$

where R, G, P, and V are periodic functions of φ_1; corresponding to case a we have $\Gamma = 1$, and to case b, $\Gamma = 0$.

3. Invariant curve existence theorem. We shall use the principle of contraction mappings in the form of /5/ to prove that the Poincaré mapping $B \mapsto B$ has an invariant curve. Let us recall the relevant conditions.

Suppose we are given the mapping T: $\bar x = f(x, \varphi)$, $\bar\varphi = \varphi + g(x, \varphi) \pmod{2\pi}$, where $\varphi \in R^m$, $x \in R^n$, $m \geqslant 1$, $n \geqslant 1$; $f(x, \varphi)$, $g(x, \varphi)$ are differentiable vector functions, 2π-periodic in $\varphi = (\varphi_1, \ldots, \varphi_m)$. We assume that T maps the ring $K = \{(x, \varphi): \|x\| < r_0, \varphi \in R^m\}$ into itself. We introduce into K the matrix or vector norm $\|(\cdot)\|_0 = \sup_{(x,\varphi)\in L} \|(\cdot)\|$, where $\|(\cdot)\|$ is the Euclidean norm. Then, under the conditions

$$
\begin{aligned}
&\|(E_m + \partial g/\partial\varphi)^{-1}\|_0 = D^{-1} \leqslant \text{const} < \infty, \quad \|\partial f/\partial x\|_0 < 1 \\
&1 - D^{-1}\|\partial f/\partial x\|_0 > 2\,[D^{-1}\|\partial g/\partial x\|_0\, \|(E_m + \\
&\quad \partial g/\partial\varphi)^{-1}\, \partial f/\partial\varphi\|_0]^{1/2} \\
&1 + D^{-1}\|\partial f/\partial x\|_0 < 2D^{-1}
\end{aligned}
\tag{3.1}
$$

where E_m is the $(m \times m)$ identity matrix, the mapping T has in K an invariant attracting m-dimensional torus.

Conditions (3.1) impose constraints on the parameter Γ and the functions R, G, P, and V. To state them, we find the transition time from A to B. Integrating the first of Eqs.(1.2), we obtain up to higher order terms in ρ:

$$t_n = \frac{1}{\mu} \ln \frac{\rho_1 (\mu + \rho_1^2)^{-1/2}}{\rho_0 (\mu + \rho_0^2)^{-1/2}}$$

where ρ_0 is the coordinate of a point in A, and the constant ρ_1 defines the position of the secant B. We take $\rho_1 \gg \varepsilon_0$. Then,

$$t_n(\rho_0) \approx \frac{1}{2\mu} \ln\left(1 + \frac{\mu}{\rho_0^2}\right) \sim \begin{cases} \rho_0^{-2}, & \mu \lesssim \rho_0^2 \\ \rho_0^{-1} \ln \rho_0^{-1}, & \mu \sim \rho_0 \end{cases} \tag{3.2}$$

Let g_μ denote the mapping $A \mapsto B$. We assume that the function N in Eqs.(1.2) identically vanishes (in the case $N \not\equiv 0$ the proof is just the same but the working is more complicated). For this model case, the mapping g_μ has the form

$$x_1 = x_0 \exp(-\lambda t_n(\rho_0)), \quad \varphi_1 = \varphi_0 + \omega\, t_n(\rho_0) \tag{3.3}$$

(the function $t_n(\rho_0)$ is given by (3.2)).

We write the superposition of mapping $h_\mu\colon g_\mu \circ f_\mu\colon B \mapsto B$

$$\begin{aligned} \bar{x}_1 &= x_0 \exp(-\lambda t_n(\bar{\rho}_0)) \\ \bar{\varphi}_1 &= \Gamma\varphi_1 + P + Vx_1 + \omega t_n(\bar{\rho}_0) \end{aligned} \tag{3.4}$$

($\bar{\rho}_0$ is defined in (2.1)). It follows at once from (3.4) that:

Lemma 1. Under the condition

$$x_0 \exp(-\lambda t_n(\bar{\rho}_0)) < \varepsilon_1 \tag{3.5}$$

the ring B is an absorbing domain of the Poincaré mapping $h_\mu(B) \in \mathrm{int}(B)$.

We will now check that conditions (3.1) hold. We shall use the subscripts H and L to indicate the maximum and minimum values of functions which depend on φ_1:

$$\max_{\varphi_1}(\cdot) = (\cdot)_H, \quad \min_{\varphi_1}(\cdot) = (\cdot)_L$$

Lemma 2. Conditions (3.1) follow from the inequalities

$$\begin{aligned} &\Gamma \neq 0, \quad D = (1 + \partial P/\partial\varphi_1 + \omega\bar{\rho}_0^{-1}(\mu + \bar{\rho}_0^2)^{-1/2}\partial R/\partial\varphi_1)_L > 0 \\ &E = (\lambda\bar{x}_1 G\delta)_H < 1 \\ &D - E > 2\,(V + \omega G\delta)_H^{1/2}\,(\lambda\bar{x}_1\delta\,(\partial R/\partial\varphi_1 + \bar{x}_1\partial G/\partial\varphi_1))_H^{1/2} \\ &D < 1; \quad \delta = \bar{\rho}_0^{-3}(1 + \mu\bar{\rho}_0^{-2})^{-1} \end{aligned} \tag{3.6}$$

where $\bar{x}_1$ is defined in (3.4).

Notice that, if the exponential term on the left-hand side of (3.5) is small, then both (3.5) and the second and third inequalities of (3.6) must hold, which implies in turn that $\bar{\rho}_0$ must be small. Recalling (3.2), this condition can be written as $\bar{\rho}_0^2 \ll \lambda$ (when $\mu \lesssim \bar{\rho}_0^2$) or as $\rho_0 \ln \rho_0 \ll \lambda$ (when $\mu \sim \bar{\rho}_0$). The first inequality of (3.6) will hold if the functions P and R are sufficiently weakly dependent on φ_1, while $\partial P/\partial\varphi_1 < 1$, $\partial R/\partial\varphi_1 \ll 1$.

Theorem 1. Under conditions (3.5) and (3.6), the mapping h_μ for sufficiently small $\mu > 0$ has a unique closed invariant curve in the ring B.

Notes. 1°. It can be shown that, in the general case $N \not\equiv 0$, conditions (3.1) certainly hold for the Poincaré mapping of system (1.1) if the λ in (3.6) is replaced by $\lambda/2$. Consequently, Theorem 1 also holds in this case. The proof is laborious and is therefore omitted.

2°. Sufficient conditions (of the type (3.6)) for an invariant curve to exist are satisfied for an open set in the class of families of systems which reveal an Andronov-Hopf reverse bifurcation, so that they are conditions of general position.

4. Corollaries (features of mode and phase portrait changes).

The trajectories of system (1.1) that pass through the invariant curve, form an attracting 2-torus. In the conditions of Theorem 1, this torus has the following features.

1°. Since $\bar{\rho}_0$ is small, see above, the torus has a strongly constricted "throat" of radius $\bar{\rho}_0$ and an outer shell which closely resembles the unstable manifold $W_0^u(\mu)$ (Fig.3,

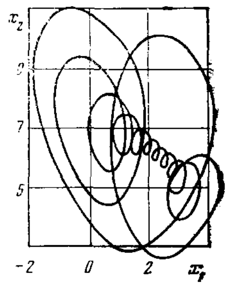

Fig.3

x_1, x_2 are the phase variables). Since R depends only weakly on φ_1 ($\partial R/\partial\varphi_1 \ll 1$), the phase portrait has almost exact axial symmetry.

2°. The time of passing through the throat is much greater than the time spent by the phase point in moving along the outer surface of the torus: $t_n \gg \lambda^{-1}$.

3°. The frequency ω of rotation about the inner turns of the throat is close to the natural frequency of the damping mode of the stationary state O in its domain of stability (with $\mu < 0$).

4°. As the parameter μ varies, the motion readjusts as follows. In the domain $\mu < 0$, the phase point is attracted to the stable CP, though even for fairly small $\mu < 0$, $|\mu|^{1/2} \lesssim R_L(\mu)$, there appears a stable 2-torus, to which part of the trajectories (which lie on "half" the unstable manifold $W_L^u(\mu)$) is attracted. The remaining trajectories in $W_L^u(\mu)$ go to CP. For $\mu = 0$, the mapping point breaks away from the small neighbourhood of CP and approaches the torus asymptotically along W_0^u. For $\mu > 0$ the torus remains the unique attractor in the domain of the phase space considered.

On varying μ in the opposite direction, we can see a hard passage to the stationary motion with $\mu = \mu_* < 0$, $|\mu_*|^{1/2} \approx R_L(\mu)$.

Notice that these laws governing the passage from the stationary to the two-frequency oscillation mode are in good agreement with those established numerically in /1, 2/ when studying convective flows in a vertical layer with wavily bent boundaries (the absence of hysteresis seems to be linked with the small undercriticality $|\mu_*| \ll 1$).

5. The case of four-dimensional phase space.

The above discussion has referred to the case when CP with $\mu = 0$, is a node in the stable set. We shall now assume that CP is a focus in W_0^s. Corresponding to this case, the minimum dimensionality n of the phase space is four.

In the neighbourhood of CP O, we can write system (1.1) as

$$\rho^{\cdot} = \mu\rho + \rho^3 + a\rho^5, \quad \varphi^{\cdot} = \omega_1$$
$$r^{\cdot} = -\lambda r + N, \quad \psi^{\cdot} = \omega_2 + M \tag{5.1}$$

where ρ, φ are the polar coordinates in the central manifold, and r, ψ in the transverse plane; N and M contain higher-order terms. As earlier, we will confine ourselves to the model situation $N, M \equiv 0$. Notice that, by following the scheme below, we can prove (under suitable conditions) the existence of a 2-torus in the general case $N, M \not\equiv 0$.

Let $A = \{r = r_0, \rho \leqslant \varepsilon_0\}$, $B = \{\rho = \rho_1, r < \varepsilon_1\}$ be the secants transverse to the trajectories of system (1.1). The constants $\varepsilon_0, \rho_1, \varepsilon_1$ and r_0 are fairly small, while $0 < \varepsilon_0 < \rho_1$, $0 < \varepsilon_1 < r_0$. We put $S_1 = W_0^u(0) \cap B$ and $S_2 = W_0^s(0) \cap A$. We assume that every semitrajectory starting at a point of S_1 reaches A. Hence it follows that, for sufficiently small μ and ε_1, there is defined the diffeomorphism f_μ: $B \mapsto A$. Under the mapping f_μ, corresponding to the point (r_1, ψ_1, φ_1) in B we have the point $(\bar\rho_0, \bar\varphi_0, \bar\psi_0)$ on the secant A:

$$\begin{aligned} \bar\rho_0 &= R(\varphi_1, \mu) + G(\varphi_1, r_1, \psi_1, \mu) r_1 \\ \bar\varphi_0 &= \Gamma_1\varphi_1 + P(\varphi_1, \mu) + V(\varphi_1, r_1, \psi_1, \mu) r_1 \\ \bar\psi_0 &= \Gamma_2'\varphi_1 + Q(\varphi_1, \mu) + W(\varphi_1, r_1, \psi_1, \mu) r_1 \\ R(\varphi_1, 0) &\geqslant R_L(0) > 0, \quad dR_L/d\mu > 0 \end{aligned} \tag{5.2}$$

where R, P, and Q periodic functions of φ_1, G, V and W are periodic functions of φ_1 and ψ_1, and $\Gamma_{1,2}$ are integers which are evaluated below.

When constructing the Poincaré local mapping g_μ: $A \mapsto B$, we note that the expression for the transition time t_n has the same form (3.2) as before. Integrating system (5.1), we obtain

$$\begin{aligned} r_1 &= r_0 \exp(-\lambda t_n(\rho_0)) \\ \varphi_1 &= \varphi_0 + \omega_1 t_n(\rho_0), \quad \psi_1 = \psi_0 + \omega_2 t_n(\rho_0) \end{aligned} \tag{5.3}$$

We write the superposition of mappings $h_\mu = g_\mu \circ f_\mu$

$$\begin{aligned} \bar r_1 &= r_0 \exp(-\lambda t_n(\bar\rho_0)) \\ \bar\varphi_1 &= \Gamma_1\varphi_1 + P + Vr_1 + \omega_1 t_n(\bar\rho_0) \\ \bar\psi_1 &= \Gamma_2\varphi_1 + Q + Wr_1 + \omega_2 t_n(\bar\rho_0) \end{aligned} \tag{5.4}$$

We have

Lemma 3. Under the condition

$$r_0 \exp(-\lambda t_n(\bar{\rho}_0)) < \varepsilon_1 \tag{5.5}$$

the secant B is an absorbing domain of the Poincaré mapping h_μ.

Note. Since the secant B is homeomorphic to the product of the 2-torus with a segment, while the flow (1.1) is built up over the Poincaré mapping h_μ, it can be shown that, when $\Gamma_1 = 1$ and condition (5.5) holds, there is an absorbent domain for system (1.1) which is homeomorphic to the product of a 3-torus with a segment.

Let us give the conditions under which an absorbent domain which is homeomorphic to the product of a circle and a 2-disc is isolated in B.

Lemma 4. Under the conditions

$$\Gamma_2 = 0, \quad 0 < \zeta_1 \leqslant Q + W\varepsilon_1 + \omega_2 t_n(\bar{\rho}_0) \leqslant \zeta_2 < 2\pi \tag{5.6}$$

there is an absorbent domain with respect to the variable ψ.

For, if we take as the absorbent domain the segment $[\bar{\zeta}_1, \bar{\zeta}_2]$, $0 < \bar{\zeta}_1 < \bar{\zeta}_2 < 2\pi$, where $\bar{\zeta}_1 < \zeta_1$ and $\bar{\zeta}_2 > \zeta_2$, then, for sufficiently small ε_1 and μ, we have $\psi_1 \in (\bar{\zeta}_1, \bar{\zeta}_2)$.

Theorem 2. By Lemmas 3 and 4, the mapping h_μ has in B an absorbent domain B_1 which is homeomorphic to the product of a circle and a 2-disc.

We will now show that conditions (3.1) hold in the domain B_1, where we understand here by x the vector (r, ψ). We have

Lemma 5. Conditions (3.1) follow from the inequalities

$$\begin{gathered}\Gamma_1 \neq 0,\ D = (1 + \partial P/\partial\varphi_1 + r_1 \partial V/\partial\varphi_1 + \omega_1 t_n'(\bar{\rho}_0)\, \partial\bar{\rho}_0/\partial\varphi_1)_L > 0 \\ E = \{[\lambda \bar{r}_1 t_n'(\bar{\rho}_0) G]_H^2 + [W + \omega_2 t_n'(\bar{\rho}_0) G]_H^2\}^{1/2} < 1 \\ D - E > 2\{[V + \omega_1 t_n'(\bar{\rho}_0) G]_H [(\partial Q/\partial\varphi_1 + r_1 \partial W/\partial\varphi_1 + \\ \omega_2 t_n'(\bar{\rho}_0)\, \partial\bar{\rho}_0/\partial\varphi_1)^2 + (\lambda \bar{r}_1 t_n'(\bar{\rho}_0)\, \partial\bar{\rho}_0/\partial\varphi_1)^2]_H^{1/2}\}^{1/2}, \quad D < 1 \\ \left(t_n'(\bar{\rho}_0) = \frac{1}{\bar{\rho}_0^3 (1 + \mu/\bar{\rho}_0^2)},\ \frac{\partial\bar{\rho}_0}{\partial\varphi_1} = \frac{\partial R}{\partial\varphi_1} + r_1 \frac{\partial G}{\partial\varphi_1}\right)\end{gathered} \tag{5.7}$$

($\bar{r}_1$ is given in (5.4)).

For (5.7) to hold, the following conditions must be satisfied: $t_n \gg \lambda^{-1}$ (so that $r_1 \ll 1$ from (5.4)); the functions G, V, and W are sufficiently small (while $G \ll 1$); and the functions R, P, and Q depend only weakly on φ_1 (while $\partial R/\partial\varphi_1$, $\partial Q/\partial\varphi_1 \ll 1$).

Theorem 3. Under conditions (5.5)-(5.7), the Poincaré mapping h_μ has in the domain B_1 a unique attracting invariant **curve**, while system (1.1) has a 2-torus.

6. Concluding remarks. 1°. In systems with a phase space of higher dimensionality there can also exist a bifurcation mechanism similar to that discussed. The only difference mathematically is that the number of "contracting" coordinates in the mapping increases, while conditions of the type (3.1) and (5.7) remain virtually unchanged (the λ in them is now the decrement of the most weakly damped mode).

2°. The first condition of (3.1) on the phase φ_1 mapping implies that points which are close in phase cannot diverge strongly in time. If we assume the contrary, i.e., that $\partial(P + \omega t_n(\bar{\rho}_0))/\partial\varphi_1 > 1$ (that the mapping $\varphi_1 \mapsto \varphi_1$ is not one-to-one), then, following /6/, we can write conditions under which there is in B a stochastic behaviour of the trajectories. The same applies to the case $n = 4$ considered in Sect.5.

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